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On the Slow Decay of $O(2)$ Correlations in the Absence of Topological Excitations: Remark on the Patrascioiu–Seiler Model

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For spin models with $O(2)$-invariant ferromagnetic interactions, the Patrascioiu–Seiler constraint is $|\arg(\underline{S}(x)) - \arg(\underline{S}(y))| \leqslant \theta_0$ for all $|x-y|=1$. It is shown that in two-dimensional systems of two-component spins the imposition of such constraints with θ_0 small enough indeed results in the suppression of exponential clustering. More explicitly, it is shown that in such systems on every scale the spin–spin correlation function obeys $\langle \underline{S}(x) \cdot \underline{S}(y) \rangle \geqslant 1/(2\,|x-y|^2)$ at any temperature, including $T=\infty$. The derivation is along the lines proposed by A. Patrascioiu and E. Seiler, with the yet unproven conjectures invoked there replaced by another geometric argument.

KEY WORDS: Continuous symmetry; Kosterlitz–Thouless transition; decay of correlations; Fortuin–Kasteleyn representation; topological charges.

1. INTRODUCTION

Two-dimensional classical spin models with continuous symmetries exhibit interesting relations between the decay of equilibrium-averaged correlations and the topological features of "typical" spin configurations. Such relations have been seen in numerous works, and at the intuitive level they can be related to elementary (free-) energy estimates. However, our understanding of this mechanism is not complete.

For instance, in studying the $O(2)$ invariant plane-rotor model on $\mathbb{Z}^2$, with $\underline{S}_x$ two-dimensional unit vectors and the interaction

$$H = -\sum_{\{x,y\}} J_{x-y} \underline{S}(x) \cdot \underline{S}(y), \qquad J_{x-y} \geqslant 0 \tag{1.1}$$

Dedicated to Oliver Penrose on the occasion of his 65th birthday.

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one can associate to each spin configuration a configuration of charges $\{q_p\}$ located on the dual lattice of plaquettes $\{p\}$ which may be viewed as representing "curl arg($\underline{S}$)" [with $\arg(\underline{S}(x))$ defined only mod 2π]. For a given plaquette, q_p is defined as the sum, along the boundary of p, of the increments of the angles $\arg(\underline{S}(x))$, with the convention that the magnitude of the increase over each pair of neighboring sites does not exceed π. The values assumed by q_p are integer multiples of 2π, and their definition makes sense regardless of H. There is evidence, present in both heuristic discussions and rigorous work on the Kosterlitz–Thouless transition[1,2] that the binding of such topological charges $\{q_p\}$ into neutral clusters is related to slow decay of spin–spin correlations (power-law instead of exponential). Nevertheless, there is no clear-cut statement (at least to the author's knowledge) that in any sense one is either sufficient or necessary for the other.

Related to this question is a striking observation made by Patrascioiu and Seiler[1] in their discussion of the mechanisms associated with the decay of correlations in $O(N)$ models. They present arguments which made it very plausible that in constrained versions of such two-dimensional $O(2)$ models, with the spins restricted to satisfy

$$|\arg(\underline{S}(x)) - \arg(\underline{S}(y))| \leqslant \theta_0 \qquad \text{whenever} \quad |x-y| = 1 \tag{1.2}$$

if θ_0 small enough, then at no temperature do the spin–spin correlations decay exponentially. Their analysis is based on percolation-theoretic and topological arguments, and is done under the assumption that some yet unproven conjectures are valid.

The main purpose of this note is to point out that the Patrascioiu–Seiler effect does indeed take place, regardless of the validity of the conjectures.[1] The result is summarized by the following statement, which provides a more explicit and assumption-free version of the one envisioned there.

We consider here models with $O(2)$ invariance, with interactions which are more general than (1.1), having the form

$$H = -\sum_{\{x,y\}} g_{x,y}(\underline{S}(x)\cdot\underline{S}(y)) \tag{1.3}$$

with monotone-increasing functions $g_{x,y}(\cdot)$ ($\uparrow$). The equilibrium averages for such system in a finite region Λ, with either the free or (when meaningful) the periodic boundary conditions, and subject to the constraints (1.2), take the form:

$$\langle f(S(\cdot))\rangle_\Lambda = \frac{\int f(S(\cdot))\, e^{-\beta H_\Lambda(S)} K(S) \prod_{x\in\Lambda} \rho_0(dS_x)}{\int e^{-\beta H_\Lambda(S)} K(S) \prod_{x\in\Lambda} \rho_0(dS_x)} \tag{1.4}$$

Here $\rho_0(\cdot)$ is the uniform measure on the circle, H is given by (1.3), and $K(S)$ is the indicator function which is 1 if the constraints (1.2) are satisfied.

Theorem 1. In a two-dimensional model of the type presented above, if the constraining angle meets the condition

$$\theta_0 \leqslant \begin{cases} \pi/4 & \text{for the cubic lattice} \\ \pi/2 & \text{for the triangular lattice} \end{cases} \tag{1.5}$$

then the equilibrium correlation function does not decay faster than $1/(2\,|x-y|^2)$ in the sense that

$$\max_{\substack{L \leqslant |x-y| \leqslant \sqrt{2}\,L \\ 0 \leqslant |x|,\, |y|,\, \leqslant L}} \langle \underline{S}_x \cdot \underline{S}_y \rangle \geqslant \frac{1}{2L^2} \tag{1.6}$$

at any temperature (including $t = \infty$) and for any finite volume Λ containing the square $[1, L] \times [1, L]$.

With few improvements—the essential one being the geometric argument given in Section 3—the proof of Theorem 1 is based on ideas of Patrascioiu and Seiler.[1] A key step is the following reduction of the spin–spin correlation to a connectivity function, which is enabled by an extension of the Fortuin–Kasteleyn construction to this model.[9, 1]

Lemma 1. In a constrained system, with θ_0 satisfying the condition of Theorem 1,

$$\mathbb{E}(\underline{S}(x) \cdot \underline{S}(y)) \geqslant \operatorname{Prob}(x \overset{P}{\Leftrightarrow} y) := \tau(x, y) \tag{1.7}$$

where $x \Leftrightarrow^P y$ denotes the event that, for the specified spin configuration $\underline{S}(\cdot)$, x and y are *-connected by a path along which the spins satisfy the "broad polar cap condition":

$$|\underline{S}(u) \cdot \underline{i}| \geqslant 1/\sqrt{2} \qquad \text{for} \quad \underline{i} = (1, 0) \tag{1.8}$$

A path is referred to as *-connected if it is connected in the sense in which the external boundary of any nearest-neighbor connected set is connected. For the rectangular lattice that means that the path's consecutive sites are either nearest neighbors or next nearest neighbors. On the triangular lattice the notion of connectedness is self-dual (i.e., *-connected = connected).

One may note that under the constraint (1.2), with θ_0 as in Theorem 1, the admissible configurations are free of topological charges, i.e., $q_p \equiv 0$. This observation raises a question about an alternative approach to this topic, on which we comment in Section 4.

2. THE CLIQUE REPRESENTATION

In this section we summarize the proof of Lemma 1, which involves mainly the Fortuin–Kasteleyn representation[4] which was extended to the systems considered here in ref. 1. Its essential features are:

1. Attention is focused on the discrete symmetry which consists of flipping just the first coordinate of the spin variables. Correspondingly, the two-component unit spin vectors $S_x = (S_{||}(x), S_\perp(x))$ are written as

$$S_x = (\sigma(x)\,|S_{||}(x)|,\, S_\perp(x)) \tag{2.1}$$

with $\sigma_x = \pm 1$ and $|S_{||}(x)| = [1 - S_\perp(x)^2]^{1/2}$. The flip R consists of the reversal of σ.

2. For specified values of the orthogonal spin components $S_\perp(\cdot)$, the conditional distribution of the Ising variables $\sigma(\cdot)$ in the constrained system is governed by a pair potential of the form

$$H = -\sum_{\{x,y\}\in B} h_{x,y}(\sigma(x), \sigma(y) \mid S_\perp(\cdot)) \tag{2.2}$$

where (i) $h_{x,y}$ is allowed to assume the value $+\infty$ (but not $-\infty$) as a way of imposing the constraint (1.2), and (ii) the sum is over the collection of bonds ($\equiv$pairs of sites) B which includes those for which $J_{x,y} \neq 0$, and also those for which the constraint (1.2) is imposed. In its dependence on $\sigma(\cdot)$ the interaction is ferromagnetic in the sense that for each specified configuration of the $S_\perp$ variables the two-body potentials satisfy

$$\begin{aligned} h_{x,y}(+,+ \mid S_\perp(\cdot)) = h_{x,y}(-,- \mid S_\perp(\cdot)) &\leqslant h_{x,y}(+,- \mid S_\perp(\cdot)) \\ &= h_{x,y}(+,- \mid S_\perp(\cdot)) \end{aligned} \tag{2.3}$$

3. The property (2.3) permits us to invoke the Fortuin–Kasteleyn "random cluster" representation of the conditional state of $\sigma(\cdot)$, condi-

[2] Clique representations have been found useful for numerical simulations (beating the critical slowing down[5] and for rigorous results[4,6] and have also been encountered in quantum systems.[7]

tioned on $S_\perp(\cdot)$. The representation may be viewed as polarizing the ferromagnetic state into a superposition of states in which any two spins are either totally independent or rigidly aligned.[2] More completely, the state is depicted as a superposition of ones in which the spins are organized into "cliques." For any specified partition the spins $\sigma(\cdot)$ are assuming uniform values on each clique, but the values for different cliques are independent and symmetrically distributed.

For this representation, the model is augmented by bond variables $\{n(x, y)\}_{\{x,y\} \in B}$ with values in $\{0, 1\}$. For a specified bond configuration $n(\cdot)$ the cliques are the clusters of sites connected by bonds with $n(x, y) = 1$. The marginal distribution of the spins, obtained by a complete integration (or summation) over $n(\cdot)$, reproduces the measure seen in (1.4).

The joint distribution of $\{\sigma(\cdot), S_\perp(\cdot), n(\cdot)\}$ can be written explicitly.[1] We shall not repeat here the somewhat standard expression, but just note that it has the following key properties.

(i) For specified values of $\{S_\perp(\cdot), n(\cdot)\}$, the conditional distribution of $\sigma(\cdot)$ has the clique structure described above.

(ii) For specified values of $\{\sigma(\cdot), S_\perp(\cdot)\}$, the conditional distribution of $n(\cdot)$ is that of the independent bond percolation model, with the probability that a given bond is occupied given by

$$\text{Prob}(n(x, y) = 1 \mid \sigma(\cdot), S_\perp(\cdot)) = \begin{cases} 0 & \text{if} \quad \Delta h \leqslant 0 \\ 1 - \exp(-\beta\, \Delta h) & \text{if} \quad \Delta h > 0 \end{cases} \tag{2.4}$$

with

$$\Delta h = h_{x,y}(-\sigma(x), \sigma(y) \mid S_\perp(\cdot)) - h_{x,y}(\sigma(x), \sigma(y) \mid S_\perp(\cdot)) \tag{2.5}$$

A key implication of the clique structure is that the spin–spin correlation function can be related to a connectivity probability. Averaging over $\sigma(\cdot)$ at given $n(\cdot)$ and, for concreteness sake, at given $S_\perp(\cdot)$, one gets

$$\mathbb{E}(\sigma(x)\, \sigma(y) \mid n(\cdot), S_\perp(\cdot)) = I[x \overset{n}{\Leftrightarrow} y] \tag{2.6}$$

where $x \Leftrightarrow^n y$ denotes the event that x and y belong to the same connected cluster, with respect to the bond configuration $\{n\}$, and $I[\text{-}]$ is the corresponding indicator function.

Next, the average over $n(\cdot)$, with its appropriate distribution conditioned on $S_\perp(\cdot)$ (whose details do not matter here) results in the equality

$$\mathbb{E}(\sigma(x)\, \sigma(y) \mid S_\perp(\cdot)) = \text{Prob}(x \overset{n}{\Leftrightarrow} y \mid S_\perp(\cdot)) \tag{2.7}$$

which can be employed for the following expression for the spin–spin correlation function:

$$\begin{aligned}\mathbb{E}(S(x)\cdot S(y)) &= 2\mathbb{E}(|S_{\parallel}(x)|\cdot|S_{\parallel}(y)|\,\sigma(x)\,\sigma(y))\\ &= 2\mathbb{E}(|S_{\parallel}(x)|\cdot|S_{\parallel}(y)|\,\mathbb{E}(\sigma(x)\,\sigma(y)\mid S_{\perp}(\cdot)))\\ &= 2\mathbb{E}(|S_{\parallel}(x)|\cdot|S_{\parallel}(y)|\,\mathrm{Prob}(x\overset{n}{\Leftrightarrow}y\mid S_{\perp}(\cdot)))\end{aligned} \tag{2.8}$$

The above equation yields the useful bound:

$$\mathbb{E}(S(x)\cdot S(y))\geqslant\mathbb{E}(\ I[\,|S_{\parallel}(x)|,\,|S_{\parallel}(y)|\geqslant 1/\sqrt{2}]\cdot\mathrm{Prob}(x\overset{n}{\Leftrightarrow}y\,|\,S_{\perp}(\cdot)\,)\ \tag{2.9}$$

which is not far from being an equality, in the sense that omitting the factor $\frac{1}{2}I[\cdot]$ the right side provides a bound in the opposite direction.

We note that Eq. (2.9) does not depend on the value of the constraining angle θ_0, and is just a reflection of the clique structure. In particular, it holds also for the unconstrained $O(2)$ model.

A notable effect of the constraint (1.2), within the random cluster extension of the model, is that for any configuration $\underline{S}(\cdot)$ in which the spin $\sigma(x)$ cannot be flipped without violating the relation (1.2) between $\underline{S}(x)$ to $\underline{S}(y)$, the conditional probability that $n(x, y)=1$ is one. [In terms of (2.4), $\Delta h=+\infty$.] In effect, a new constraint is dynamically generated any bond with

$$|\arg(S(x))-\arg(S(y))|\leqslant\theta_0\leqslant|\arg(RS(x))-\arg(S(y))| \tag{2.10}$$

is (a.s.) occupied. [$R\underline{S}(x)$ being the spin obtained by reflecting $S_{\parallel}(x)$]. Hence

$$\mathrm{Prob}(x\overset{n}{\Leftrightarrow}y\mid S_{\perp}(\cdot))\geqslant I[x\overset{C}{\Leftrightarrow}y] \tag{2.11}$$

where $x\Leftrightarrow^{C}y$ denotes the event that, for the specified spin configuration $S(\cdot)$, x and y are connected by a path along which at each step the condition (1.9) is satisfied.

When θ_0 is small enough, as spelled out in Theorem 1, $x\Leftrightarrow^{C}y$ is guaranteed to occur whenever $\underline{S}$ is an allowed configuration in which x and y are *-connected by a path along which the spins take values only within the polar caps defined by (1.8). That is,

$$I[x\overset{C}{\Leftrightarrow}y]\geqslant I[x\overset{P}{\Leftrightarrow}y] \tag{2.12}$$

The combination of the three statements (2.9), (2.11), and (2.12) proves Lemma 1.

3. THE GEOMETRIC ARGUMENT

The lower bound for the correlation function stated in Theorem 1 is now a consequence of the following geometric considerations, stated initially for the cubic lattice. Let us focus on the spin configuration within the rectangle $[1, L] \times [1, L]$. The boundary of this rectangle consists of the four intervals: B_1 (top),..., B_4 (ordered consecutively).

In each configuration $\underline{S}(\cdot)$, there either is a path connecting a pair of sites on the top and bottom faces, $x \in B_1$ and $y \in B_3$, in the sense denoted here by $x \Leftrightarrow^P y$, or else the $\Leftrightarrow^P$ connected cluster of B_1 is separated from B_3. In the latter case, the boundary of that cluster forms a connected path (in the regular sense) which reaches from a point on the left face, $u \in B_2$, to a point on the right face, $v \in B_4$, along which the spins fail (1.8), i.e., satisfy the complementary condition:

$$|S_{\perp}(u)| \geqslant 1/\sqrt{2} \tag{3.1}$$

Let this dual event be denoted $u \Leftrightarrow^{P\perp} v$.

By rotation invariance, and the fact that the *-connectedness required for $u \Leftrightarrow^{P\perp} v$ is less stringent than the connectedness needed for $u \Leftrightarrow^P v$, we have

$$\bar{\mathrm{P}}\mathrm{rob}(u \overset{P\perp}{\Leftrightarrow} v) \geqslant \mathrm{Prob}(u \overset{P}{\Leftrightarrow} v) = \tau(u, v) \tag{3.2}$$

The dichotomy spelled out above means that one of the events in the union always occurs, and thus

$$\mathrm{Prob}\left(\bigcup_{x \in B_1, y \in B_3} \{x \overset{P}{\Leftrightarrow} y\} \cup \bigcup_{u \in B_2, y \in B_4} \{u \overset{P\perp}{\Leftrightarrow} v\} \right) = 1 \tag{3.3}$$

Since the number of events listed above is $2L^2$, the probability of at least one of them exceeds $1/(2L^2)$. For the pair of corresponding sites, say $\{x, y\}$, we have (using Eq. (1.7))

$$\langle \underline{S}(x) \cdot \underline{S}(y) \rangle \geqslant \tau(x, y) \geqslant \frac{1}{2L^2} \tag{3.4}$$

Since x and y are on opposite faces of the boundary of $[1, L] \times [1, L]$, (3.4) establishes the slow decay described in Theorem 1.

The above argument is robust, and for other lattices requires only a trivial adjustment: the sites x, y, u, and v need not lie on the boundary of $[1, L] \times [1, L]$, but instead they will be within distance one from it.

4. DISCUSSION

It was noted by A. Polyakov that the result of Theorem 1 fits well with the folklore that exponential decay requires the presence of unbound charges. It would be of interest to see a derivation of a bound along these lines. The following observation may be of relevance. For a bond $b \equiv \{x, y\}$ let

$$\Theta_{x,y} = \arg(\underline{S}(x)) - \arg(\underline{S}(y)) + k2\pi \tag{4.1}$$

with $k = 0, \pm 1$ chosen so that $|\Theta_{x,y}| \leqslant \pi$. Using Stokes' theorem, in the absence of the topological charges (defined in the Introduction), we have

$$\sum_{b \in \partial[1,L] \times [1,L]} \Theta_b = \sum_{P \in [1,L] \times [1,L]} q_P = 0 \tag{4.2}$$

That implies that the collection of the bond variables $\{\Theta_b\}$ are strongly correlated. Can one extract from that a statement like Theorem 1? Further insight along these lines may also be useful in the discussion of other phenomena related to the Kosterlitz–Thouless transition.

It may also be of interest to point out that in the absence of the topological charges (i.e., when $q_P \equiv 0$) there is a one-to-one correspondence between the angles $\theta(x) = \arg(\underline{S}(x))$ and real variables $\varphi(x)$ satisfying $\varphi(0) = \theta(0)/\theta_0$, and

$$\varphi(x) - \varphi(y) = \Theta_{x,y}/\theta_0 \tag{4.3}$$

[without the corrections seen in (4.1)]. In terms of the variables $\varphi(\cdot)$, the constraint (1.2) on $\theta(\cdot)$ (at $T = \infty$) corresponds to the "hammock" potential:

$$V(\varphi(x) - \varphi(y)) = \begin{cases} 0 & |\varphi(x) - \varphi(y)| \leqslant 1 \\ +\infty & |\varphi(x) - \varphi(y)| > 1 \end{cases} \tag{4.4}$$

Under the correspondence $\theta(\cdot) \Leftrightarrow \varphi(\cdot)$, Theorem 1 acquires the following form.

Theorem 2. Let $V(\cdot)$ be a monotone-increasing function on $[0, \infty)$ which diverges for $|z| > 1$. Then, in $(d = 2)$-dimensional lattice systems of real-valued variables $\{\varphi(x)\}$ with the distribution

$$\rho(d\varphi) = \left\{ \exp\left[-\beta \sum_{\langle x,y \rangle} V(|\varphi(x) - \varphi(y)|) \right] \right\} \delta(\varphi(0))\, \Pi_x\, d\varphi(x)/\text{Norm} \tag{4.5}$$

the following bound holds, at any $\beta \geqslant 0$:

$$\int \cos(t[\varphi(x) - \varphi(0)])\, \rho(d\varphi) \text{ “}\geqslant\text{” } \frac{1}{2\,|x|^2} \qquad \text{for all} \quad |t| < \theta_0 \tag{4.6}$$

where “$\geqslant$” is to be read in the sense made explicit in (1.6).

The intuitive explanation of (4.6) is that the long-distance behavior of $(\varphi(x) - \varphi(0))$ is similar to that seen in Gaussian models with short-range elastic forces, for which (in $d = 2$ dimensions) $\langle |\varphi(x) - \varphi(0)|^2 \rangle$ grows, but at only a logarithmically slow rate. An interesting discussion, and a number of results on the rate of growth of $|\varphi(x) - \varphi(0)|$ in two-dimensional models (which, however, do not include the hammock potential), can be found in ref. 8.

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